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A characterization of graphs of girth eight or more with exactly two sizes of maximal independent sets

A. Finbow and B. Hartnell
Saint Mary's University, Halifax, Canada

C. Whitehead
Goldsmiths' College, University of London, London, UK

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Abstract

In 1970, Plummer introduced the notion of a well-covered graph as one in which every maximal independent set is a maximum. Here, we study graphs in which there are exactly two sizes of maximal independent sets. A characterization of such graphs is obtained for graphs of girth eight or more.

1. Introduction

Plummer [9] defined a *well-covered* graph as one in which every maximal independent set of vertices is a maximum. Whereas, the problem of determining a maximum independent set of vertices is well known to be very difficult for an arbitrary graph, for a well-covered graph any maximal independent set will suffice. In this article we consider graphs, denoted by M_2, which have exactly two sizes of maximal independent sets. For instance, the 6-cycle, the 8-cycle as well as the path on 5 vertices are graphs in M_2. If M_i represents graphs with precisely i sizes of maximal independent sets, then M_1 is the collection of well-covered graphs. Finbow and Hartnell [6] characterized the well-covered graphs of girth 8 or more as being those in which every vertex was either a leaf or had exactly one leaf as a neighbour (with the single exceptional graph K_1). In this paper, we characterize those graphs of girth 8 or more in M_2. Since a graph is in M_2 if and only if it has one connected component in M_2 and

Correspondence to: C. Whitehead, Department of Mathematical Sciences, Goldsmiths' College, New Cross, London SE14 6NW, England.

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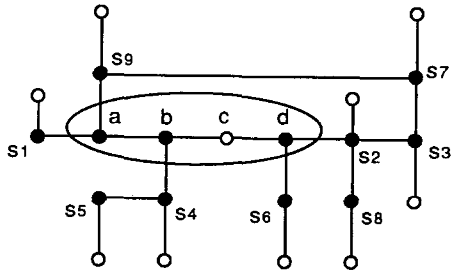

Fig. 1.

all others well covered, we henceforth restrict our attention to connected graphs. The reader is referred to [1–4, 7, 8, 10–12] for further results on M_1 and to [5] for a generalization.

A vertex of degree one will be called a *leaf* and a vertex adjacent to a leaf will be called a *stem*. Whenever a stem is denoted by s_i and has a single leaf as a neighbour, we denote that leaf as l_i. A vertex which is neither a stem nor a leaf will be called *leafless*.

L is used to denote the set of all leaves in a graph and SL (single leaves) the set of all leaves whose neighbouring stems have exactly one leaf attached.

A cycle of length i is denoted by C_i and P_i represents a path on i vertices.

For any set S of vertices in a graph G, $N[S]$ denotes the set of all vertices either in S or adjacent to a vertex in S.

A straightforward, but extremely useful, observation (Lemma 2.1) is that for any independent set S of vertices of a graph G belonging to M_2, the resulting graph $G-N[S]$ (and hence each of its components) must be in M_1 or M_2. For instance, Fig. 1 illustrates a graph in M_2, having maximal independent sets of size 10 and 11, where letting S be the set of all leaves, $G-N[S]$ is P_4. In the next section it is shown that if we consider S as the set L or SL, then there are very limited possibilities for the components in $G-N[S]$. Our general strategy will be to first consider the possible structure of a component in either $G-N[L]$ or $G-N[SL]$ and then to examine the various possibilities for G if there are several components.

2. Results

The following lemma, with $k=2$, proves to be very useful in the characterization.

Lemma 2.1. *If $G\in M_k$ and I is an independent set of vertices then $G-N[I]\in M_i$ for some $i\leqslant k$.*

Proof. Assume $G-N[I]$ has maximal independent sets, say $I_1, I_2, \ldots, I_{k+1}$ of $k+1$ different sizes. But then $I_1\cup I, I_2\cup I, \ldots, I_{k+1}\cup I$ would be maximal independent sets of $k+1$ different sizes in G, a contradiction. $\square$

We now consider the restrictions on $G-N[L]$. Lemma 2.2 is a vital observation in our considerations.

Lemma 2.2. *If v is a leafless vertex in a graph G in M_2 and G is of girth 8 or more, then v has at most two neighbours which are leafless.*

Proof. Consider G in M_2 where G is of girth 8 or more and assume some vertex v is leafless and has at least three neighbours, say a, b and c which are also leafless.

Let I_1 be the set of all vertices which are both at distance 2 from v and distance 3 from a, b and c. Let I_2 be the set of all vertices, which are both at distance 3 from v and distance 2 from exactly one of a, b and c. Note that $I_1 \cup I_2$ is independent (by the girth restriction). Extend $I_1 \cup I_2$ to a maximal independent set, say M, of $G-N[v]$.

However, both $M \cup \{v\}$ and $M \cup \{a, b, c\}$ are maximal independent sets of G and of the same parity.

Now consider I_3, which consists of a neighbour of a other than v and all vertices which are both at distance 3 from v and distance 2 from exactly one of b and c. Observe that $I_1 \cup I_3$ is independent and extend $I_1 \cup I_3$ to a maximal independent set, say M', of $G-N[v]$. Hence $M' \cup \{v\}$ and $M' \cup \{b, c\}$ are maximal independent sets of G and of different parity. This is a contradiction. □

Corollary 2.3. *If $G \in M_2$ and is of girth 8 or more, then each component of $G-N[L]$ is K_1, a path or a cycle.*

Proof. By Lemma 2.2, the vertices of $G-N[L]$ are of degree 0, 1 or 2. □

Corollary 2.4. *Say G is of girth 8 or more and no vertex of G is a leaf. Then $G \in M_2$ if and only if G is one of C_8, C_9, C_{10}, C_{11} or C_{13}.*

Proof. By Lemma 2.2, G is forced to be a cycle. It is straightforward to verify that membership in M_2 and the girth restriction result in G being isomorphic to one of C_8, C_9, C_{10}, C_{11} or C_{13}. □

Corollary 2.5. *If $G \in M_2$ and is of girth 8 or more, then a component of $G-N[L]$ that is well covered must be isomorphic to exactly one of K_1, P_2 and P_4.*

Proof. This follows directly from the girth restriction, Corollary 2.3 and [6]. □

Each of the graphs in Fig. 2 is an example of a graph $G \in M_2$, where $G-N[L]$ has a well-covered component. In the case that $SL=L$ and $G-N[L]$ has only one component, note that all graphs with this property can be derived from these examples by the following operation (repeated as many times as necessary). Join a new stem which has a single leaf to any subset of the black vertices as long as the girth restriction is maintained. This new stem is itself black in the resulting graph.

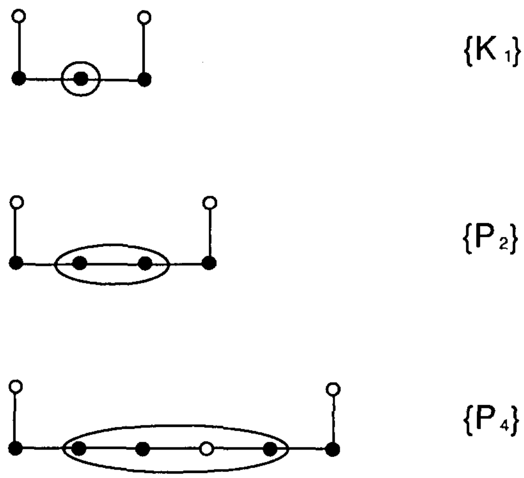

Fig. 2.

For instance, the graph in Fig. 1 can be obtained from the third graph in Fig. 2 by first joining the stem s_3 and its leaf to that graph and then joining the stem s_4 and its leaf to the graph just created and so on. We also observe that, as indicated in Fig. 2, not all of the vertices of P_4 can be black else we could form a maximal independent set of a third size by including a neighbouring stem of each vertex in P_4 (by the girth condition such stems would be independent).

Finally, we also note that all such graphs are in fact in M_2. For the first pair of graphs of Fig. 2, we have maximal independent sets of size $|L|, |L|+1$ and for the last $|L|+1$ and $|L|+2$.

Lemma 2.6. *Let G be a graph in M_2 of girth* 8 *or more and suppose that G contains at least one leaf. Then a component of $G-N[L]$ that is in M_2 must be isomorphic to exactly one of P_3, P_6, P_8, C_8, C_9 or C_{11}.*

Proof. By Corollary 2.3, we need only consider paths and cycles.

First observe that the only possible paths in M_2 are P_3, P_5, P_6, P_7, P_8 and P_{10}. Our strategy is to show that if $G-N[L]$ has a component isomorphic to any of P_5, P_7 or P_{10}, then G contains maximal independent sets of at least three different sizes, contradicting the assumption that G is in M_2. In each case, we start by constructing a maximum independent set of the largest cardinality. Note that the set L is itself an independent set of G. Further, for any leaf x with stem s, exactly one of x and s must be included in any maximal independent set of G. However, since the set S of stems is not necessarily independent and in any case $|S| \leqslant |L|$, we can always construct a maximum independent set in G by taking the set union of L with maximum independent sets in each component of $G-N[L]$.

Assume $P_5=[abcde]$ is a component in $G-N[L]$. Let s_1 (resp., s_2) be a stem adjacent to a (resp., e) in G. Extend $L\cup\{a,c,e\}$ to a maximum independent set I_1 of G. $I_2=I_1\cup\{b,d\}-\{a,c,e\}$ is also a maximal independent set. Now extend $\{s_1,s_2,c\}$ to a maximal independent set I_3. As $|I_3|<|I_2|<|I_1|$, G cannot belong to M_2. Hence, P_5 cannot be a component of $G-N[L]$.

Next assume $P_7=[abcdefg]$ is a component in $G-N[L]$. Let s_1 be a stem adjacent to a in G. Observe that s_1, c and f are independent by the girth condition.

Extend $L\cup\{a,c,e,g\}$ to a maximum independent set I_1 of G. $I_2=I_1\cup\{b,d,f\}-\{a,c,e,g\}$ is a maximal independent set. Now extend $\{s_1,c,f\}$ to a maximal independent set I_3. However, $|I_3|<|I_2|<|I_1|$ which contradicts G belonging to M_2. Thus, P_7 cannot be a component of $G-N[L]$.

Assume $P_{10}=[abcdefghij]$ is a component of $G-N[L]$. Let s_1 (resp., s_2) be a stem adjacent to a (resp., j).

If $s_1=s_2$ or s_1 and s_2 are adjacent, extend $\{s_1,c,f,i\}$ to a maximal independent set I_3 of G.

If s_1 and s_2 are independent, and s_1 not joined to h then either $\{s_1,h,e,s_2,b\}$ or $\{s_1,h,e,s_2,c\}$ are independent. Extend this independent set to a maximal independent set I_3 of G. If s_1 is joined to h but s_2 not joined to c, then extend $\{s_1,s_2,c,f,i\}$ to a maximal independent set I_3 of G. In the event both s_1 is joined to h and s_2 is joined to c, extend $\{s_2,b,e,h\}$ to a maximal independent set I_3 of G.

Now extend $L\cup\{a,c,e,g,j\}$ to a maximum independent set I_1 of G and note that $I_2=I_1\cup\{b,d,f,i\}-\{a,c,e,g,j\}$ is a maximal independent set. But $|I_3|<|I_2|<|I_1|$ and hence P_{10} cannot be a component of $G-N[L]$.

We now turn our attention to cycles as possible components. Since the only cycles in M_2 are C_8, C_9, C_{10}, C_{11} and C_{13} (for girth $\geqslant 8$), we need only consider these.

If $C_{10}=(abcdefghij)$ were a component in $G-N[L]$, let s_1 be a stem adjacent to a, say, in G. However, extending $L\cup\{a,c,e,g,i\}$ to a maximum independent set I_1 of G, noting that $I_2=I_1\cup\{b\}-\{a,c\}$ is a maximal independent set and extending $\{s_1,c,f,i\}$ to a maximal independent set I_3 of G, we have $|I_3|<|I_2|<|I_1|$, a contradiction.

Similarly, if $C_{13}=(abcdefghijklm)$ were a component in $G-N[L]$, let s_1 be a stem adjacent to a, say, in G. Again extend $L\cup\{a,c,e,g,i,k\}$ to a maximum independent set I_1 of G. $I_2=I_1\cup\{d\}-\{c,e\}$ is a maximal independent set. However, extending $\{s_1,c,f,i,l\}$ to a maximal independent set I_3 of G we have $|I_3|<|I_2|<|I_1|$. Hence, C_{13} is not a component of $G-N[L]$.

It is easy to verify that C_8, C_9 and C_{11} are possible depending on the adjacencies of stems. □

In the case that $SL=L$ and $G-N[L]$ has exactly one component, say K (noting that K has maximal independent sets of sizes x and $x+1$), we observe that G itself has maximal independent sets of size $|L|+x$ and of size $|L|+x+1$. There are various possibilities for which vertices of K are joined to stems. However, we must ensure that regardless of which stems are in a maximal independent set, that set includes at least x vertices of K.

For instance, not all three vertices of P_3 can be adjacent to stems (as those stems would be independent by the girth condition) else G would have maximal independent sets of sizes $|L|, |L|+1$ and $|L|+2$.

Each of the graphs in Fig. 3 is an example of a graph $G \in M_2$, where $G-N[L]$ has a component in M_2. In the case that $SL=L$ and $G-N[L]$ has exactly one component, we note that all graphs with this property can be derived from these examples by

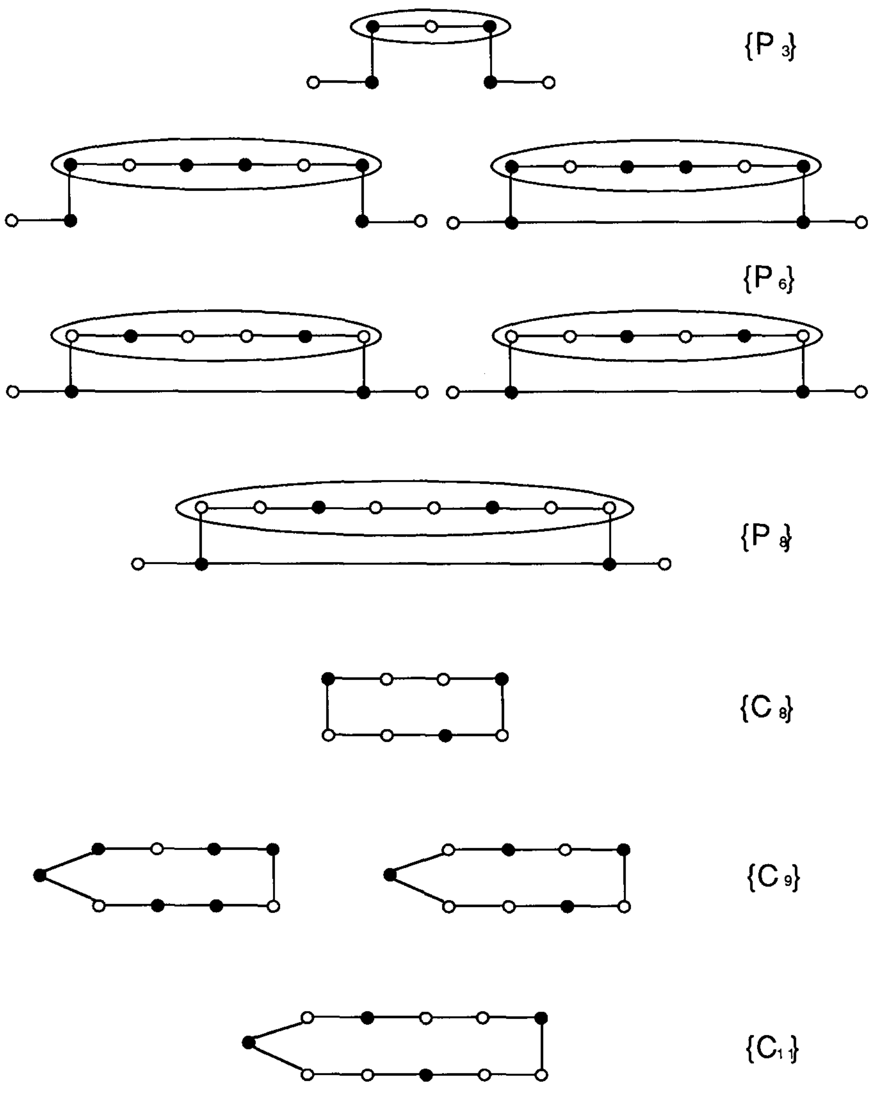

Fig. 3.

the following operation (repeated as many times as necessary). Join a new stem which has a single leaf to any subset of the black vertices as long as the girth restriction is maintained. This new stem is itself black in the resulting graph.

Although we do not give all the details, the following observations are helpful in the verification of the possible arrangements of the black vertices. In general, we observe that there cannot be three consecutive black vertices (as G would then contain a maximal independent set of size $|L|+x-1$) nor we can have two blacks at one end of a path (noting that the other end is adjacent to at least one stem). For P_8, we note that the end vertices cannot be adjacent to independent stems and also that the vertex next to the end of P_8 cannot be black. Observing that there cannot be two consecutive blacks we are forced into the configuration given. For C_8, there cannot be two consecutive black vertices nor two which are at distance 4. For C_9, consider the cases in which there are two adjacent blacks and in which there are not. In the case of C_{11}, there cannot be two adjacent black vertices nor two which are at distance 4.

We now examine the various possibilities for there to be several components in $G-N[L]$. First, we consider the case in which there are at least two well-covered components.

Lemma 2.7. *If $G\in M_2$ and is of girth at least 8, and H_1 and H_2 are well-covered components of $G-N[L]$, then either $\{H_1,H_2\}\cong\{P_2,P_4\}$ or $\{H_1,H_2\}\cong\{P_2,P_2\}$. Furthermore, if $P_4=[abcd]$ and $P_2=[ef]$, then, in G, a, d, e, f and at least one of b and c are of degree two and these vertices form a 10-cycle $(s_1abcds_2s_4fes_3)$, where s_1,s_2,s_3 and s_4 are stems in G. If $H_1\cong P_2=[ab]$ and $H_2\cong P_2=[cd]$, then for some choice of $x\in\{a,b\}$ and $y\in\{c,d\}$, x and y are each of degree two and have exactly one stem as a neighbour and these stems are adjacent.*

Proof. Consider $G\in M_2$ with girth at least 8 and let H_1 and H_2 be well-covered components of $G-N[L]$. Assume $H_1\cong K_1=\{v\}$. Say $H_2\cong K_1=\{w\}$. The vertex v has at least two neighbours, say s_1 and s_2, which are stems and w has at least two neighbours, say s_3 and s_4, which are stems. At least one of s_1 and s_2, say s_1, is distinct from both s_3 and s_4 and also independent of w, s_3 and s_4 else the girth condition is violated. But extend L to a maximum independent set I_1 of G, $\{s_1\}$ to as large a maximal independent set as possible, say I_2, and $\{s_1,s_3\}$ to a maximal independent set I_3. Then $|I_3|<|I_2|<|I_1|$, which is a contradiction.

Assume $H_2\cong P_2=[ab]$. Again v has at least two neighbours, say s_1 and s_2, in G which are stems and a has a neighbouring stem, say s_3, and b has a neighbouring stem, say s_4. By girth, either s_1 or s_2, say s_1, is distinct and independent from both s_3 and s_4 as well as a and b. But extend L to a maximum independent set I_1 of G, $\{s_1\}$ to as large a maximal independent set, say I_2, as possible and $\{s_1,s_3,s_4\}$ to a maximal independent set I_3. However, $|I_3|<|I_2|<|I_1|$, a contradiction.

Assume $H_2\cong P_4=[abcd]$. Let two neighbouring stems of v be s_1 and s_2 and s_3 be a stem adjacent to a. Either s_1 or s_2, say s_1, must be independent of and distinct from s_3 and c (by the girth condition). Extend L to a maximum independent set I_1, $\{s_1\}$ to as large a maximal independent set, say I_2, as possible, and $\{s_1, s_3, c\}$ to a maximal independent set I_3. But $|I_3|<|I_2|<|I_1|$ which is a contradiction.

Hence, neither H_1 nor H_2 can be isomorphic to K_1 when there are at least two well-covered components in $G-N[L]$.

Next assume both H_1 and H_2 are isomorphic to P_4, say $H_1=[abcd]$ and $H_2=[efgh]$. Let the stems s_1, s_2, s_3 and s_4 be adjacent to a, d, e and h, respectively. If s_1 is adjacent to either e or h, say e, then extend L to a maximum independent set I_1, $\{s_1, c\}$ to as large a maximal independent set, say I_2, as possible, and $\{s_1, c, g\}$ to a maximal independent set, say I_3. Again $|I_3|<|I_2|<|I_1|$, a contradiction.

In the event s_1 is adjacent to neither e nor h, we note that at least one of $\{s_1, s_3, g\}$ or $\{s_1, s_4, f\}$ is independent, say $\{s_1, s_3, g\}$ (by the girth condition). Choose I_1 and I_2 as before and extend $\{s_1, s_3, c, g\}$ to a maximal independent set I_3. Then $|I_3|<|I_2|<|I_1|$ as above.

Hence H_1 and H_2 cannot both be isomorphic to P_4.

Assume $H_1 \cong P_4=[abcd]$ and $H_2 \cong P_2=[ef]$. Let s_1, s_2, s_3 and s_4 be stems adjacent to a, d, e and f, respectively.

If either $\{s_1, s_3, s_4, c\}$ or $\{s_2, s_3, s_4, b\}$ is independent, say the former, extend L to a maximum independent set, say I_1. Then extend $\{s_1, c\}$ to as large a maximal independent set, say I_2, as possible and extend $\{s_1, s_3, s_4, c\}$ to a maximal independent set, say I_3. But $|I_1|>|I_2|>|I_3|$ which is a contradiction. If either s_1 or s_2, say s_1, is such that $s_1=s_3$ or $s_1=s_4$, say $s_1=s_3$, then the same maximal independent sets force a contradiction.

Hence, the only possibility is for s_1 and s_3 to be adjacent and s_2 and s_4 to be adjacent, where a, d, e and f are all of degree two. If b and c both had stems, say s_5 and s_6, respectively, as neighbours, then extend $\{s_1, s_2, s_5, s_6\}$ to a maximal independent set I'_3. Then the sets I_1, I_2 and I'_3 are of three different sizes, a contradiction. Hence, at least one of b and c is of degree two.

Finally, we consider $H_1 \cong P_2=[ab]$ and $H_2 \cong P_2=[cd]$. If there are stems s_1, s_2, s_3 and s_4 adjacent to a, b, c and d, respectively, where the stems are independent, extend L to a maximum independent set I_1, $\{s_1, s_2\}$ to as large a maximal independent set as possible, say I_2, and $\{s_1, s_2, s_3, s_4\}$ to a maximal independent set, say I_3. In the event $s_1=s_3$ say, extend $\{s_1, s_2, s_4\}$ to a maximal independent set say I_3. However, these are of three different sizes, which is a contradiction.

Hence, at least one of the pairs a and c, a and d, b and c or b and d, say a and c, have the following property: a and c are each of degree two and have exactly one stem as a neighbour and these stems are adjacent. □

The three graphs in the top row of Fig. 4 illustrate Lemma 2.7 when there are exactly two components in $G-N[L]$, while the more complicated cases of three or more are shown in the rest of the figure.

We note that there cannot be three components where one is P_4. For assume there is another $P_2=[gh]$, besides $P_2=[abcd]$ and $P_2=[ef]$ arranged as in Fig. 4. Considering $[abcd]$ and $[gh]$, we see that neither g nor h can be adjacent to s_1 or s_2 (as

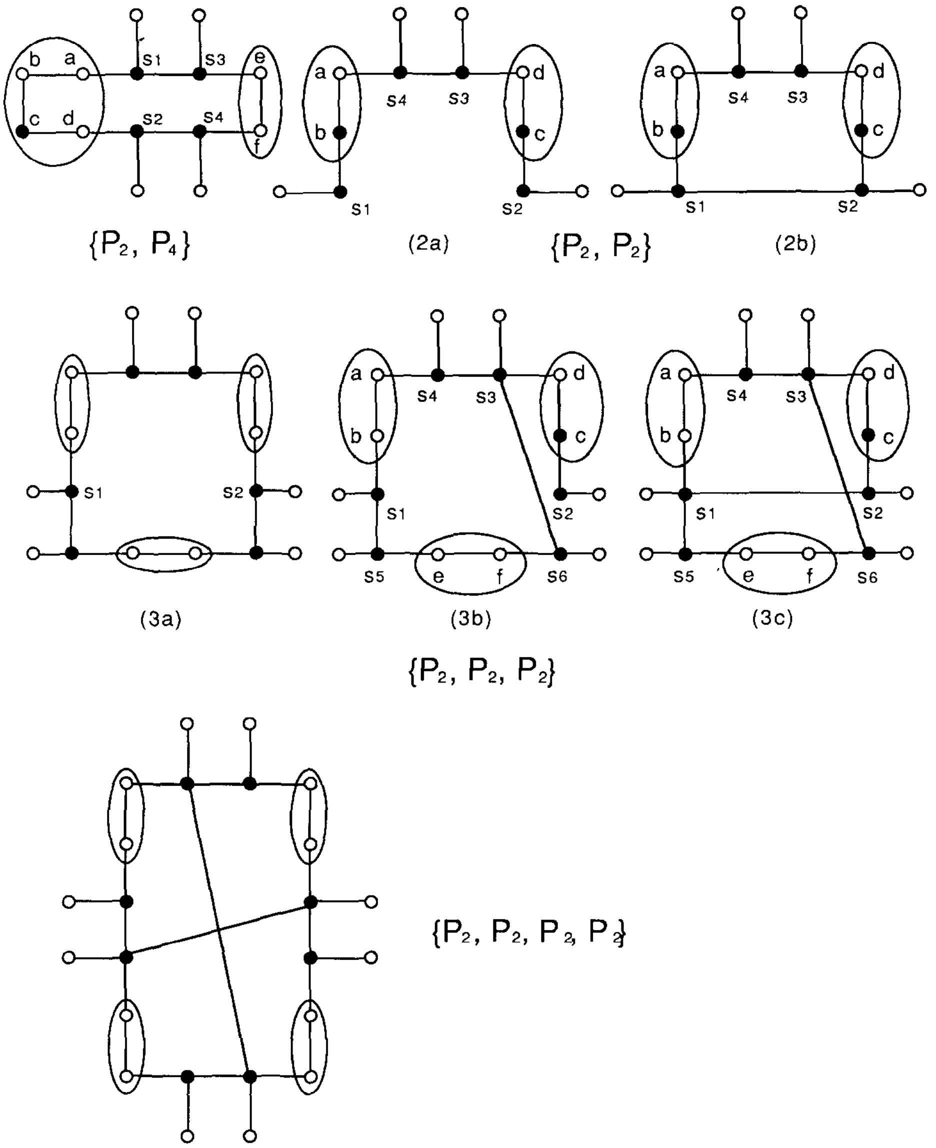

Fig. 4.

there must be two stems between a P_4 and a P_2). However, considering [ef] and [gh], one of g and h, say g, must be adjacent to a stem, s_5 say, which in turn is adjacent to s_3 (without loss of generality). Since g must be at distance 3 from either a or d this forces either s_5 and s_1 to be adjacent (girth 3) or s_5 and s_2 to be adjacent (girth 6).

Next, consider the possibility of three P_2's as components of $G-N[L]$. The graphs in (2a) and (2b) illustrate the arrangement for [ab] and [cd]. Whether s_1 and s_2 are adjacent or not, letting $[ef]$ be the third P_2 we note that neither e nor f can be adjacent to any of the stems s_1, s_2, s_3 or s_4. For if e, say, were adjacent to s_1 or s_4, say s_4, then consider $[ab]$ and $[ef]$. Since e and a are at distance two, f and one of a and b must be at distance three (to satisfy Lemma 2.7). However, this would violate the girth restriction. Hence, e is adjacent to another stem, say s_5, and f to s_6, say.

In the case that s_1 and s_2 are adjacent, considering $[cd]$ and $[ef]$, we may, without loss of generality, assume that s_3 and s_6 are adjacent. Considering $[ab]$ and $[ef]$, the lemma and girth requirement force s_1 and s_5 to be joined (graph (3c)).

Now consider the case that s_1 and s_2 are not joined. If either s_5 or s_6 is adjacent to either s_3 or s_4, say s_6 to s_3, then consider $[ab]$ and $[ef]$. To satisfy Lemma 2.7, one of the pairs s_4 and s_6, s_1 and s_6, s_1 and s_5 or s_4 and s_5 must be adjacent. The girth restriction forces s_1 and s_5 to be adjacent (graph (3b)).

The other possibility is that none of the edges s_3s_5, s_3s_6, s_4s_5 nor s_4s_6 is present. Considering $[cd]$ and $[cf]$, s_2 must be adjacent to either s_5 or s_6. Now examine $[ab]$ and $[ef]$. Either s_1 and s_5 are adjacent (graph (3a)) or s_1 and s_6 are joined (graph (3b)).

Suppose there is a fourth P_2 in $G-N[L]$. Noting that it cannot be adjacent to an existing stem, it is straightforward to verify that the arrangement shown in Fig. 4 is the only one possible. Similarly, one can check that it is impossible to have more than four P_2's.

To summarize, each of the graphs in Fig. 4 is an example of a graph $G \in M_2$ where $G-N[L]$ has several well-covered components. Observe that all connected graphs with this property can be derived from these examples by the operation (repeated as many times as necessary) of joining a new stem which has a single leaf to any subset of the black vertices as long as the girth restriction is maintained. This new stem is itself black in the resulting graph.

Next, we consider the possibility of $G-N[L]$ containing two components H_1 and H_2, with H_1 being well covered and $H_2 \in M_2$.

Lemma 2.8. *If $G \in M_2$ and is of girth 8 or more, then $G-N[L]$ cannot have a well-covered component H_1 as well as a component $H_2 \in M_2$.*

Proof. Let $G \in M_2$ and be of girth 8 or more. Assume $H_1, H_2 \in G-N[L]$ where H_1 is well covered and $H_2 \in M_2$. First, we show that H_2 is not a cycle. By Lemma 2.6, H_2 can only be C_8, C_9 or C_{11}.

Let J_1 and J_2 be two maximal independent sets of H_2 of different sizes with $|J_1| > |J_2|$. Extend $L \cup J_1$ to a maximum independent set, say I_1, of G and let $I_2 = (I_1 - J_1) \cup J_2$. Observe that $|I_1| = |I_2| + 1$.

If $H_1 \cong K_1 = \{v\}$ or $H_1 \cong P_4 = [abcd]$, let s_1 be a stem of G adjacent to v or a. Select a maximal independent set of minimum size, say J_3, of H_2 such that s_1 is not adjacent to any vertex of J_3. This is always possible since by the girth condition s_1 is adjacent to at most one vertex of H_2. Now extend $J_3 \cup \{s_1\}$ (if $H_1 = \{v\}$) or $J_3 \cup \{s_1, c\}$ (if $H_1 = [abcd]$) to a maximal independent set, say I_3, of G. However, $|I_1| > |I_2| > |I_3|$ which is a contradiction.

If $H_1 \cong K_2 = [ab]$, let s_1 and s_2 be stems adjacent to a and b, respectively. By the girth condition, there are at most two vertices in H_2 adjacent to s_1 or s_2 and hence one can still select a maximal independent set, say J_3, of minimum size in H_2 in such a way that $J_3 \cup \{s_1, s_2\}$ is an independent set. Again extend $J_3 \cup \{s_1, s_2\}$ to a maximal independent set, say I_3, of G. Once more, $|I_1| > |I_2| > |I_3|$. Hence, H_2 is not a cycle.

If H_2 is a path, then by Lemma 2.6 the only choices are P_3, P_6 or P_8. Let J_1 and J_2 be maximal independent sets of H_2 such that $|J_1| = |J_2| + 1$ and extend $L \cup J_1$ to a maximum independent set, say I_1, of G and let $I_2 = (I_1 - J_1) \cup J_2$.

Say $H_2 \cong P_3 = [abc]$. We have already seen that there is no stem adjacent to b (see Fig. 3). Now let s_1 be a stem adjacent to v (if $H_1 = \{v\}$) or d (if $H_1 = [defg]$). However, extend $\{s_1, b\}$ (if $H_1 = \{v\}$) or $\{s_1, b, f\}$ (if $H_1 = [defg]$) to a maximal independent set, say I_3, or G. However, $|I_3| < |I_2| < |I_1|$, a contradiction. If $H_1 \cong P_2 = [de]$, let s_1 and s_2 be stems adjacent to d and e, respectively. Extend $\{s_1, s_2, b\}$ to a maximal independent set, say I_3, which again is smaller than I_1 and I_2.

Say $H_2 \cong P_6 = [abcdef]$. Let s_1 be a stem adjacent to v (if $H_1 = \{v\}$) or to g (if $H_1 = [ghij]$). If s_1 is not adjacent to b nor to e, then extend $\{s_1, b, e\}$ (if $H_1 = \{v\}$) or $\{s_1, b, e, i\}$ (if $H_1 = [ghij]$) to a maximal independent set I_3 in G. If s_1 is adjacent to e, say, then let s_2 be a stem adjacent to f and extend $\{s_1, s_2, b, d\}$ (if $H_1 = \{v\}$) or $\{s_1, s_2, b, d, i\}$ (if $H_1 = [ghij]$) to a maximal independent set I_3 in G. If $H_1 \cong P_2 = [gh]$, let s_1 and s_2 be stems adjacent to g and h, respectively. If $\{s_1, s_2, b, e\}$ is independent, then extend $\{s_1, s_2, b, e\}$ to a maximal independent set, say I_3, of G. If $\{s_1, s_2, b, e\}$ is not independent, then assume, without loss of generality, that s_1 and b are adjacent. Let s_3 be a stem adjacent to a. Observe that either $\{s_1, s_2, s_3, c, e\}$ or $\{s_1, s_2, s_3, s_3, c, f\}$ is independent, say the former. However, extend $\{s_1, s_2, s_3, c, e\}$ to a maximal independent set, say I_3, of G. In all cases, $|I_3| < |I_2| < |I_1|$, which is a contradiction.

Finally, say $H_2 \cong P_8 = [abcdefgh]$. As we have seen, there is no stem adjacent to any of the vertices b, d or g (see Fig. 3). Let s_1 be a stem adjacent to v (if $H_1 = \{v\}$) or to i (if $H_1 = [ijkl]$). Extend $\{s_1, b, d, g\}$ (if $H_1 = \{v\}$) or $\{s_1, b, d, g, k\}$ (if $H_1 = [ijkl]$) to a maximal independent set, say I_3 of G. If $H_1 \cong P_2 = [ij]$, let s_1 and s_2 be stems adjacent to i and j, respectively. Extend $\{s_1, s_2, b, d, g\}$ to a maximal independent set, say I_3, of G. In each case, $|I_3| < |I_2| < |I_1|$, which is a contradiction. Hence H_2 cannot be a path either. □

We next show $G - N[SL]$ cannot have two components in M_2.

Lemma 2.9. *If $G \in M_2$ and I is an independent set of vertices of G, then $G - N[I]$ can have at most one component which is not well covered.*

Proof. Consider $G \in M_2$ and let I be an independent set of vertices of G. Assume that H_1 and H_2 are both in $G - N[I]$ and in M_2. Let J_1 and J_2 be maximal independent sets of H_1 with $|J_1| > |J_2|$ and J_3 and J_4 be maximal independent sets of H_2 with $|J_3| > |J_4|$. However, extend $I \cup J_1 \cup J_3$ to a maximum independent set, say I_1, of

G and let $I_2=(I_1-J_1)\cup J_2$. Also let $I_3=(I_2-J_3)\cup J_4$. Then $|I_1|>|I_2|>|I_3|$, which is impossible. $\square$

We now show that almost all stems must have exactly one leaf attached.

Lemma 2.10. *If $G\in M_2$ and is of girth eight or more, then there can be at most two stems with more than one leaf attached, and if there are two they must be adjacent and have the same number of leaves as neighbours.*

Proof. Let $G\in M_2$ and be of girth eight or more. Assume there are three stems, say s_1, s_2 and s_3, each with at least two leaves as neighbours. Let the leaves attached to s_1 be L_1, to s_2 be L_2 and to s_3 be L_3. At least one pair, say s_1 and s_3, of stems must be independent since there are no 3-cycles. If s_1 and s_3 were in different components of $G-N[SL]$ such components could not be well covered, but this would violate Lemma 2.9. Consider the component, say H, of $G-N[SL]$ containing s_1 and s_3. Extend $L_1\cup L_3$ to a maximum independent set of H, say I_1. Extend $\{s_1\}\cup L_3$ to as large a maximal independent set, say I_2, of H as possible. Finally, extend $\{s_1, s_3\}$ to a maximal independent set, say I_3, of H. However, $|I_1|>|I_2|>|I_3|$, which violates $H\in M_2$. Hence, there can be at most two stems with more than one leaf attached.

Assume that there are two stems, say s_1 and s_2, with leaves L_1 and L_2, respectively, attached. By the preceeding paragraph, these stems must be adjacent. Assume that $|L_1|\neq|L_2|$. Let X be the set of vertices in G which are either at distance 2 from s_1 and distance 3 from s_2 or at distance 3 from s_1 and distance 2 from s_2. By the girth condition, X is independent. Extend $L_1\cup L_2\cup X$ to a maximal independent set, say I_1, of G.

Denote $I_1\cup\{s_1\}-L_1$ by I_2 and $I_1\cup\{s_2\}-L_2$ by I_3. However, I_1, I_2 and I_3 are three maximal independent sets of G of different sizes, a contradiction. Hence $|L_1|=|L_2|$. $\square$

In the next three lemmas, we establish the conditions under which $G\in M_2$, where G is of girth 8 or more, can have one or two stems with more than one leaf attached.

Lemma 2.11. *Let $G\in M_2$ and be of girth 8 or more. If G has a stem, say s_1, with more than one leaf attached, then s_1 cannot have a leafless neighbour.*

Proof. Let $G\in M_2$ and be of girth at least 8. Assume G has a stem, say s_1, with L_1 the set of leaves attached to s_1, where $|L_1|\geqslant 2$ and w is a leafless neighbour of s_1 such that $w\notin L_1$. Let X be the set of all vertices which are either at distance 2 from s_1 and distance 3 from w or at distance 3 from s_1 and distance 2 from w. Extend X to a maximal independent set, say I, of the graph induced by $V(G)-\{s_1, w\}-L_1$. Both $I\cup\{s_1\}$ and $I\cup L_1\cup\{w\}$ are maximal independent sets of G and differ by $|L_1|$ in size. Next, let Y be the set of all vertices at distance 2 from s_1 and extend Y to a maximal independent set, say J, of the graph induced by $V(G)-\{s_1\}-L_1$. Both $J\cup\{s_1\}$ and

$J \cup L_1$ are maximal independent sets of G, but differ by $|L_1|-1$ in size. Hence $G \notin M_2$. □

Lemma 2.12. *Let $G \in M_2$ and be of girth 8 or more. G has two adjacent stems, say s_1 and s_2, with more than one leaf attached, say L_1 and L_2, respectively, where $|L_1|=|L_2|$, only if every other vertex in G is either a leaf or is a stem with exactly one leaf as a neighbour.*

Proof. Let $G \in M_2$ and be of girth 8 or more. Say G has two adjacent stems, say s_1 and s_2, with leaves L_1 and L_2 attached, respectively, where $|L_1|=|L_2|=M \geqslant 2$. We wish to show that $G-N[SL]$ has precisely one component containing the stems s_1 and s_2 and their leaves. By Lemma 2.9, any other component, H_1, of $G-N[SL]$ must be well covered.

Let s_3 be a stem adjacent to v, if $H_1=\{v\}$, or to a, if $H_1=[abcd]$, or s_3 and s_4 be stems adjacent to a and b, respectively, if $H_1=[ab]$. By the girth condition, at least one of s_1 and s_2, say s_1, is independent of s_3 (in the first two cases) or s_3 and s_4 (in the last case). Hence extend $L_1 \cup L_2$ to a maximum independent set, say I_1, of G and then $\{s_1\} \cup L_2$ to as large a maximal independent set, say I_2, as possible of G. Finally, extend $L_2 \cup \{s_1, s_3\}$(if $H_1=\{v\}$), $L_2 \cup \{s_1, s_3, c\}$ (if $H_1=[abcd]$) or $L_2 \cup \{s_1, s_3, s_4\}$ (if $H_1=[ab]$) to a maximal independent set, say I_3, of G. However, $|I_1|>|I_2|>|I_3|$ a contradiction. □

Lemma 2.13. *Let $G \in M_2$ and be of girth 8 or more. If G has exactly one stem, say s_1, with more than one leaf attached, then let H_2 be the component containing s_1 and its leaves in $G-N[SL]$. If there is another component, say H_1, in $G-N[SL]$ then H_2 is s_1 and its leaves L_1 where $|L_1|=2$ and $H_1 \cong K_2=[ab]$ where either a or b, say a, is of degree two and adjacent to a stem which is also adjacent to s_1.*

Proof. Let $G \in M_2$ and be of girth $\geqslant 8$. Say G has exactly one stem, say s_1, with more than one leaf, say L_1, attached. Assume $G-N[SL]$ has at least two components, say H_1 and H_2, where H_2 contains the graph induced by $L_1 \cup \{s_1\}$. By Lemma 2.11, we know H_2 has no other vertices and by Lemma 2.9, H_1 must be well covered.

If $H_1 \cong K_1=\{v\}$ or $H_1 \cong P_4=[abcd]$, there must be a stem, say s_2, adjacent to v (respectively, a or d, say a) that is not adjacent to s_1 (by the girth condition). Extend $SL \cup L_1$ to a maximum independent set, say I_1, of G and let $I_2=I_1 \cup \{s_1\}-L_1$. Then extend $\{s_1, s_2\}$ (if $H_1=\{v\}$) or $\{s_1, s_3, c\}$ (if $H_1=[abcd]$) to a maximal independent set, say I_3, of G. However, $|I_1|>|I_2|>|I_3|$ which is a contradiction.

If $H_1 \cong P_2=[ab]$, then if s_2 and s_3 were stems adjacent to a and b, respectively, where $\{s_1, s_2, s_3\}$ was independent, then extend $\{s_1, s_2, s_3\}$ to a maximal independent set I_3 which yields a similar contradiction with I_1 and I_2. Hence, s_2, say, is adjacent to s_1 and a is of degree two.

Let I_1 and I_2 be as before and extend $\{s_2, s_3\}$ to a maximal independent set, say I_3. In order for two of these three sets to be of the same size, L_1 must be of size two. □

Each of the graphs in Fig. 5 is an example of a graph $G \in M_2$ where there is a stem with more than one leaf attached. We observe that all graphs with this property can be derived from these examples by the operation indicated before (repeated as often as required). Join a new stem which has a single leaf to any subset of the black vertices as long as the girth restriction is maintained. This new stem is itself black in the resulting graph.

We conclude by summarizing the characterization.

Theorem 2.14. *Let G be a connected graph of girth 8 or more. $G \in M_2$ if and only if exactly one of the following holds*:

(1) *G has no leaf. Then G is one of C_8, C_9, C_{10}, C_{11} or C_{13}.*

(2) *G has exactly two stems, say s_1 and s_2, with more than one leaf attached, say L_1 and L_2, respectively. Then $|L_1| = |L_2|$, s_1 and s_2 are adjacent and every other vertex in G is either a leaf or is a stem with exactly one leaf as a neighbour.*

(3) *G has exactly one stem, say s_1, with more than one leaf, say L_1, attached. One of three subcases must occur (see Fig. 5).*

(i) *$G - N[SL]$ is simply one component, namely, s_1 and its leaves L_1.*

(ii) *$G - N[SL]$ consists of two components, say H_1 and H_2. H_2 is s_1 and its leaves L_1 where $|L_1| = 2$ and $H_1 \cong K_2 = [ab]$ where either a or b is of degree two and adjacent to a stem which is also adjacent to s_1.*

(iii) *$G - N[SL]$ consists of three components, say H_1, H'_1 and H_2. H_2 is as in* (ii) *and $H_1 = [a_1 b_1]$ and $H'_1 = [a'_1 b'_1]$. Similar to* (ii) *a_1 and a'_1 say, are each of degree two and each adjacent to a distinct stem which is adjacent to s_1. Furthermore, b_1 and b'_1 are each*

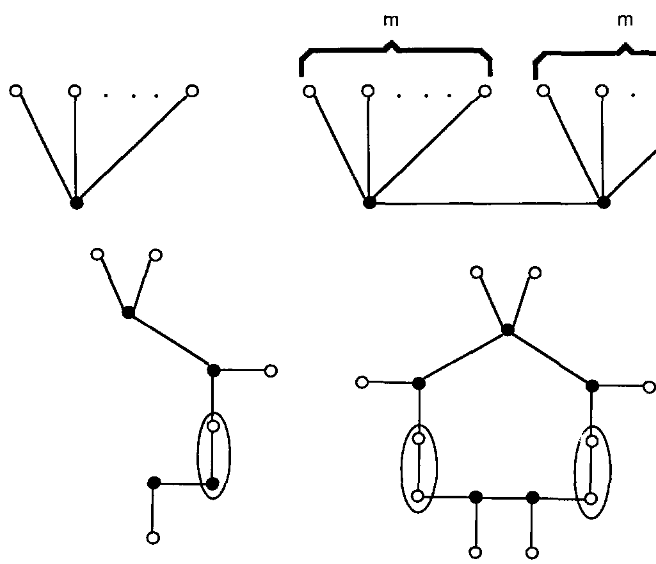

Fig. 5.

of degree two and each adjacent to a stem, say s_3 and s_4, respectively, where s_3 and s_4 are adjacent.

(4) *There are no stems with more than one leaf attached and $G-N[SL]$ has exactly one component. Then that component is $K_1, P_2, P_4, P_3, P_6, P_8, C_8, C_9$ or C_{11} and the possible adjacencies to stems (each with a single leaf) are as indicated in Figs.* 2 *and* 3.

(5) *There are no stems with more than one leaf attached and $G-N[SL]$ has several components. If any two of these components are called H_1* and *H_2 then either $\{H_1, H_2\} \cong \{P_2, P_4\}$ or $\{H_1, H_2\} \cong \{P_2, P_2\}$. Furthermore, if $P_4 = [abcd]$ and $P_2 = [ef]$, then, in G, a, c, d, e and f are all of degree two and are part of a* 10*-cycle $(s_1 abcd s_2 s_4 ef s_3)$ where s_1, s_2, s_3 and s_4 are stems in G. If $H_1 \cong P_2 = [ab]$ and $H_2 \cong P_2 = [c, d]$, then for some choice of $x \in \{a, b\}$ and $y \in \{c, d\}$, x and y are each of degree two and have exactly one stem as a neighbour and these stems are adjacent. G is either one of the graphs indicated in Fig.* 4 *or can be derived from one of them by the following operation (repeated as often as necessary). Join a new stem which has a single leaf to any subset of the black vertices as long as the girth restriction is maintained. This new stem is itself black in the resulting graph.*

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